System Dynamics and Mechanical Vibration

If you have a smart project, you can say "I'm an engineer"

Lecture 1

Staff boarder

Dr. Mohamed Saber Sokar

Dr. Mostafa Elsayed Abdelmonem

System Dynamics and Mechanical Vibration

- **Lecture aims:**
  - Formulate the equations of motion of two-degree-of-freedom systems
  - Identify the mass, damping, and stiffness matrices from the equations of motion

Two Degree-of-Freedom Systems

1. Model problem
   - Matrix form of governing equation
   - Special case: Undamped free vibrations
   - Examples
2. Transformation of coordinates
   - Inertially & elastically coupled/uncoupled
   - General approach: Modal equations
   - Example
3. Response to harmonic forces
   - Model equation
   - Special case: Undamped system

Two-Degree-of Freedom Systems

[k], and [c], [m] are called the *stiffness* , *damping*, and *mass matrices* respectively,

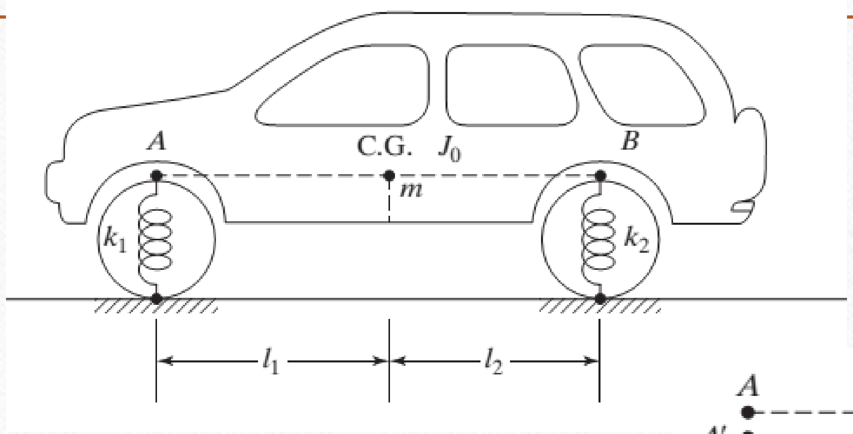
$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$

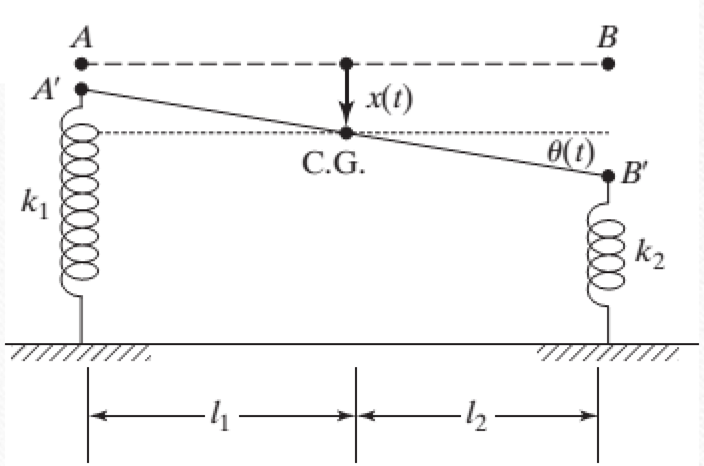
$\vec{x}(t)$ and $\vec{f}(t)$ are called the *displacement* and *force vectors*, respectively.

Two-Degree-of Freedom Systems

Thus the system has one point mass *m* and two degrees of freedom, because the mass has two possible types of motion (translations along the *y* and *x* directions). The general rule for the computation of the number of degrees of freedom can be stated as follows

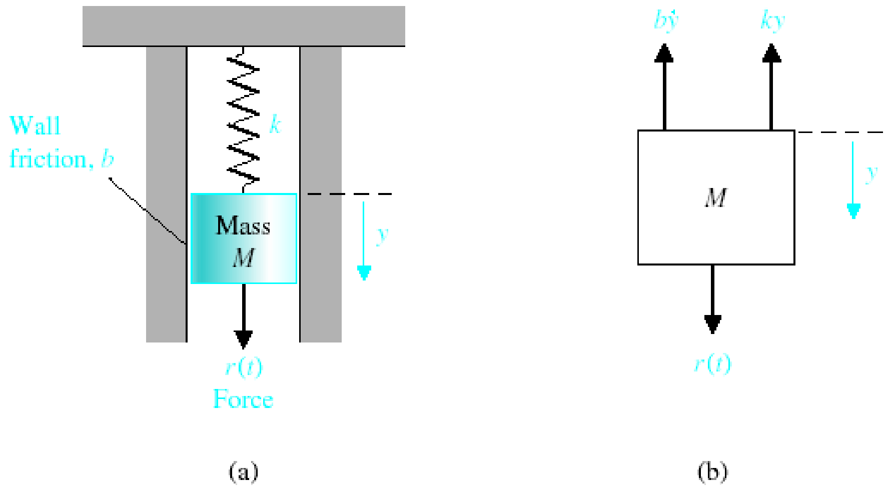
$$\begin{array}{c}\text{Number of}\\ \text{degrees of freedom}\\ \text{of the system}\end{array} = \begin{array}{c}\text{Number of masses in the system}\\ \times\ \text{number of possible types}\\ \text{of motion of each mass}\end{array}$$





Modeling of Mechanical System

- **Spring - Mass - Damper**



(a) Spring-mass-damper system.
(b) Free-body diagram.

$$M\cdot\frac{d^2}{dt^2}y(t) + b\cdot\frac{d}{dt}y(t) + k\cdot y(t) = r(t)$$

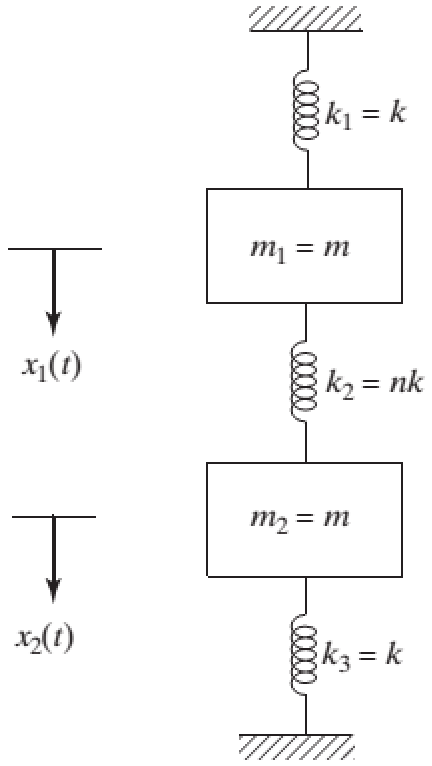
Two-Degree-of Freedom Systems

Equations of motion:

$$m_1\ddot{x}_1(t) + (k_1 + k_2)x_1(t) - k_2x_2(t) = 0$$

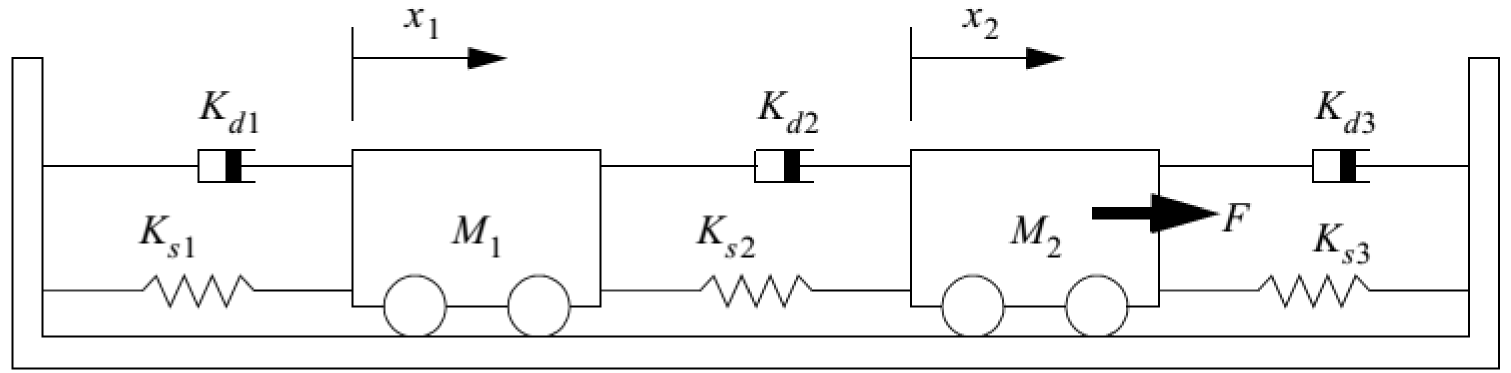
$$m_2\ddot{x}_2(t) - k_2x_1(t) + (k_2 + k_3)x_2(t) = 0$$

We are interested in knowing whether m_1 and m_2 can oscillate harmonically with the same frequency and phase angle but with different amplitudes. Assuming that it is possible to have harmonic motion of m_1 and m_2 at the same frequency ω and the same phase angle ϕ,

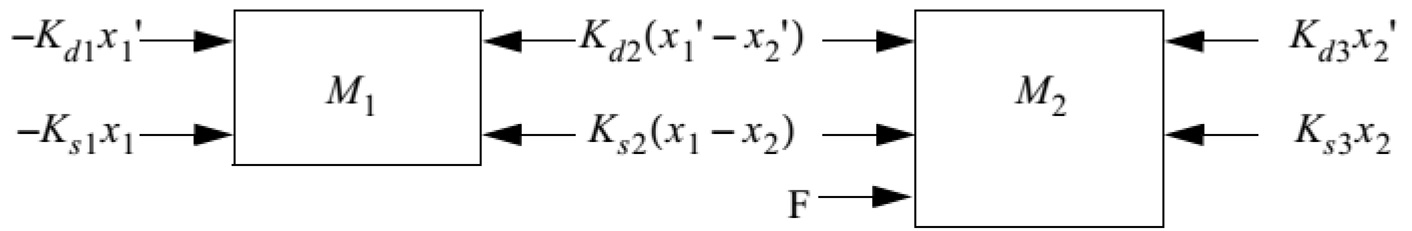


Modeling of Mechanical system

Mathematical Models for the Schematic

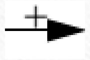


- Free Body Diagram FBD

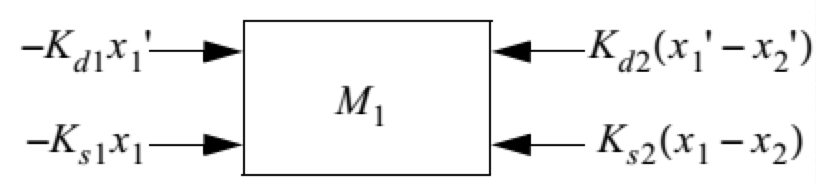


Modeling of Mechanical system

Write equation of motion: Two degree of freedom

Assume X1 >X2 positive direction of motion 

- For mass(1)



$$\sum F = -K_{d1}x_1' - K_{s1}x_1 - K_{d2}(x_1' - x_2') - K_{s2}(x_1 - x_2) = M_1 x_1''$$

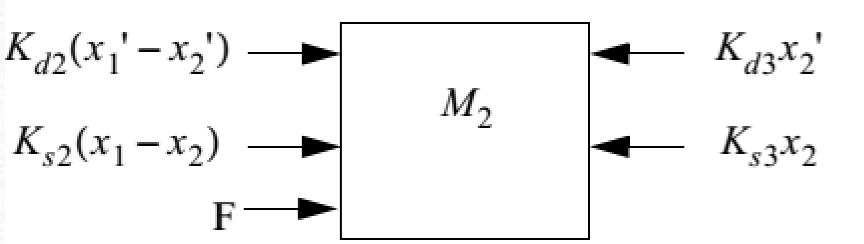
$$x_1''(M_1) + x_1'(K_{d1} + K_{d2}) + x_1(K_{s1} + K_{s2}) + x_2'(-K_{d2}) + x_2(-K_{s2}) = 0$$

Modeling of Mechanical system

Write equation of motion: Two degree of freedom

Assume X1 >X2 positive direction of motion $\xrightarrow{+}$

- For mass(2)



$$\sum F = K_{d2}(x_1'-x_2') + K_{s2}(x_1-x_2) + F - K_{d3}x_2' - K_{s3}x_2 = M_2 x_2''$$

$$x_2''(M_2) + x_2'(K_{d2}+K_{d3}) + x_2(K_{s2}+K_{s3}) + x_1'(-K_{d2}) + x_1(-K_{s2}) = F$$

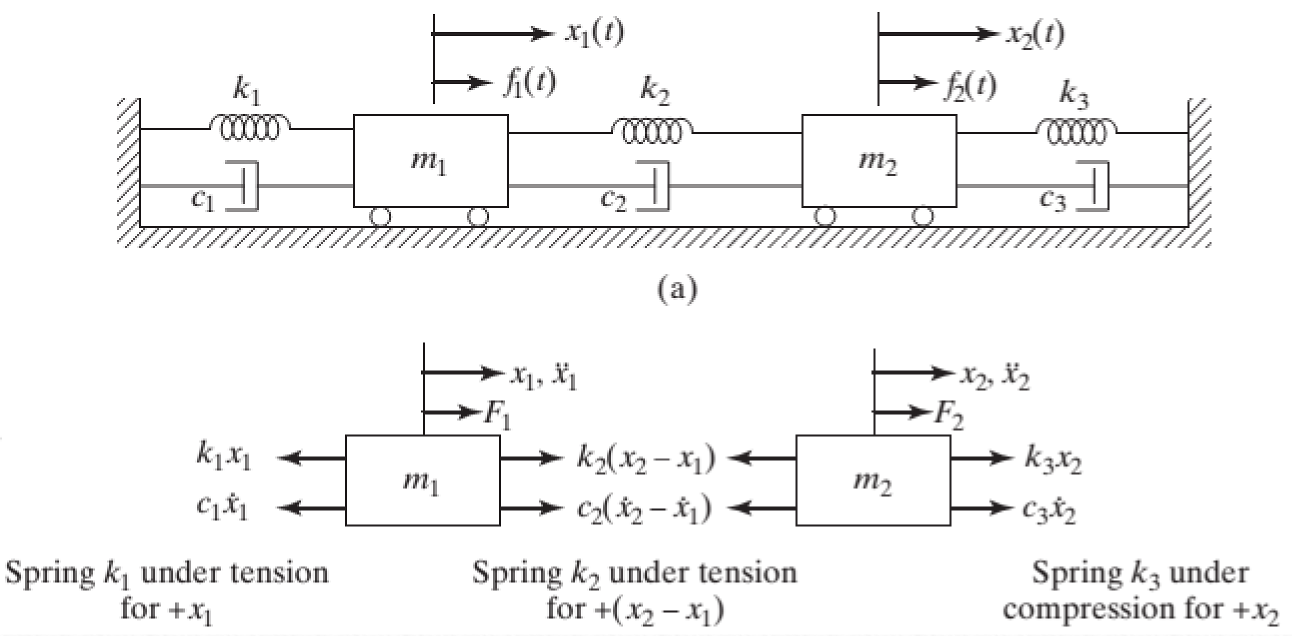
Two-Degree-of Freedom Systems

Equations of motion:

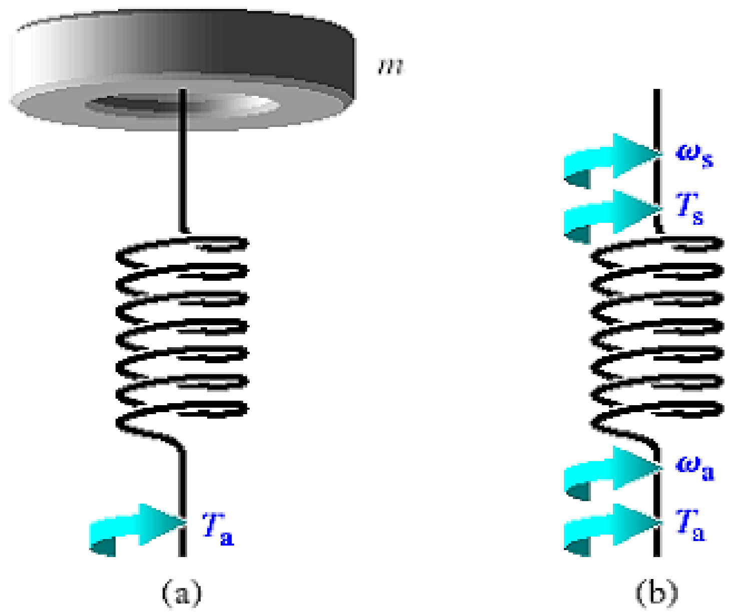
$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$



Modeling of Mechanical System



(a) Torsional spring-mass system.
(b) Spring element.

$T_a(t) - T_s(t) = 0$

$T_a(t) = T_s(t)$

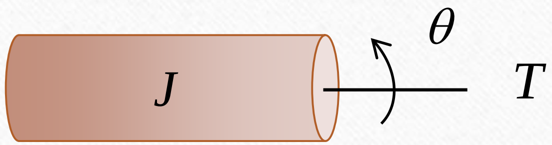
$\omega(t) = \omega_s(t) - \omega_a(t)$

$T_a(t)$ = through - variable

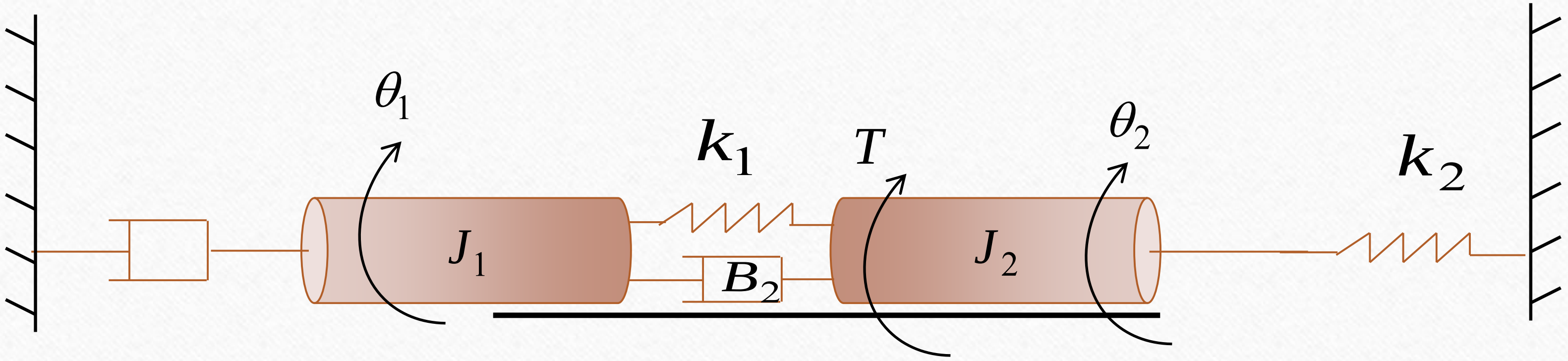
angular rate difference = across-variable

Example

Moment of Inertia



$$T = J\ddot{\theta}$$



Two-Degree-of Freedom Systems

Equations of motion:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$

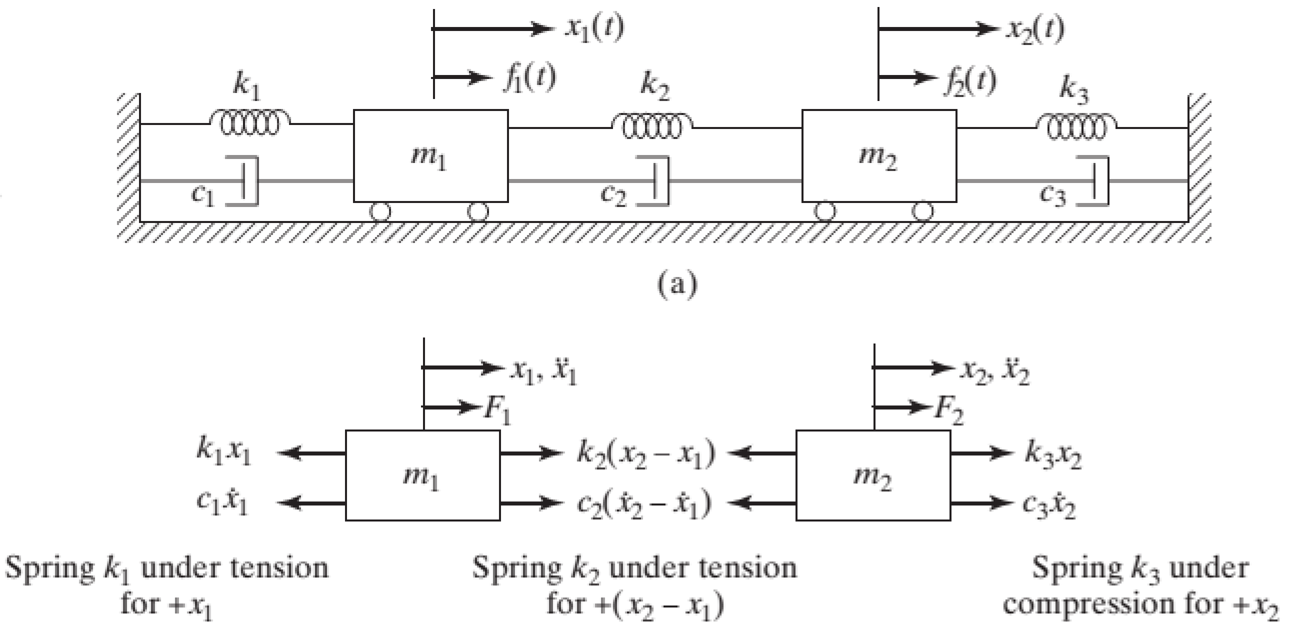
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2$$

$$[m]\,\ddot{\vec{x}}(t) + [c]\,\dot{\vec{x}}(t) + [k]\,\vec{x}(t) = \vec{f}(t)$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$

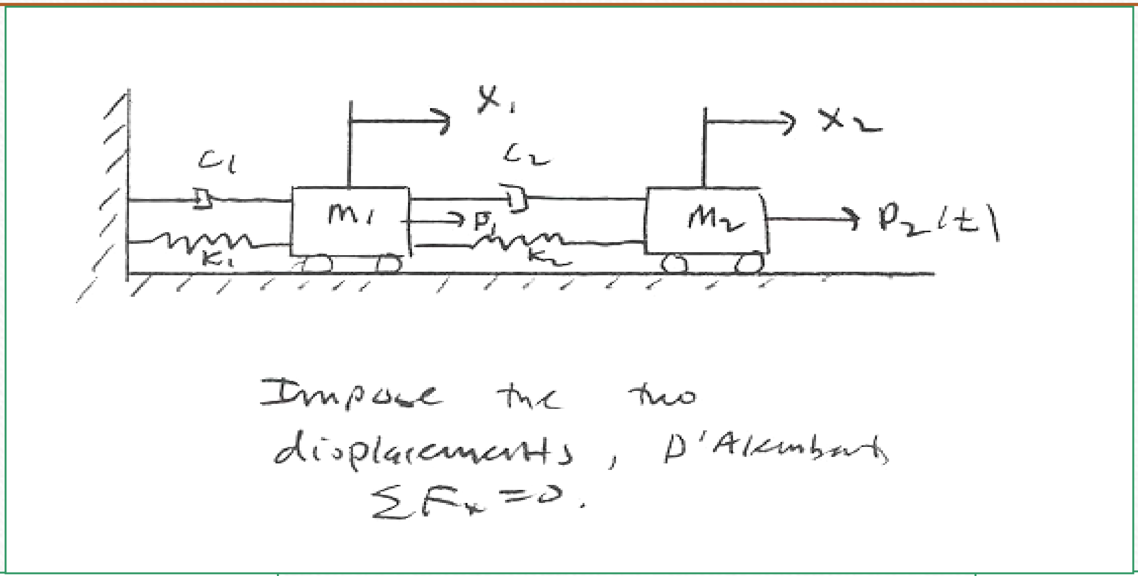
$$[c] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}$$

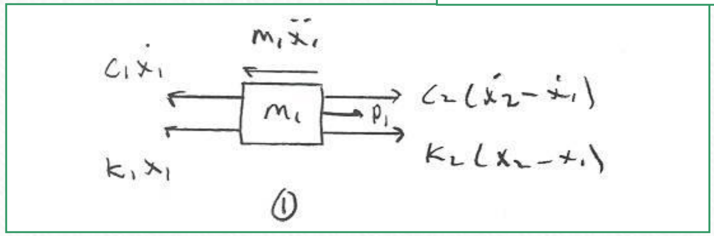
$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

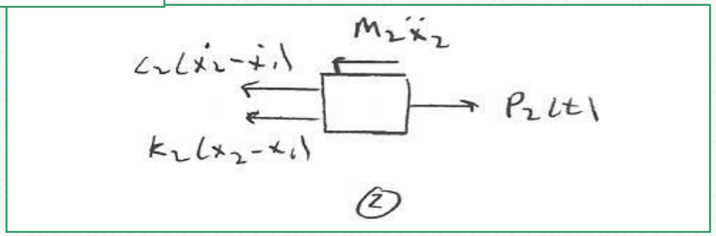


Two-DOF model problem

<u>Matrix form of governing equation:</u>

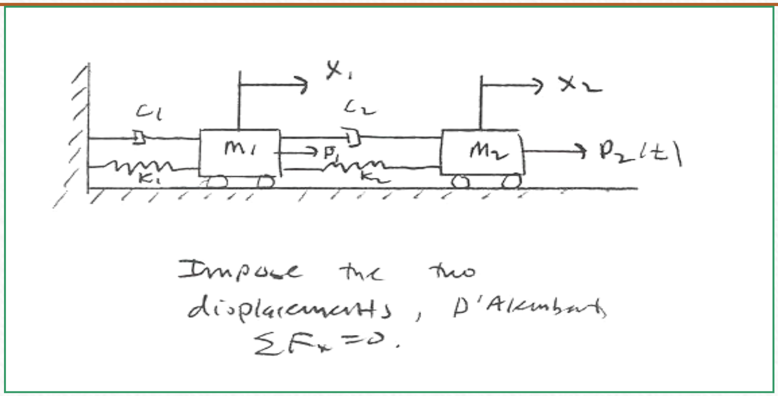


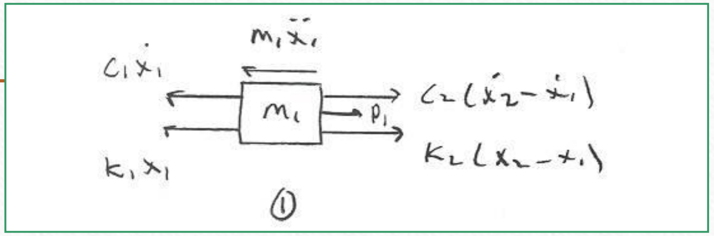


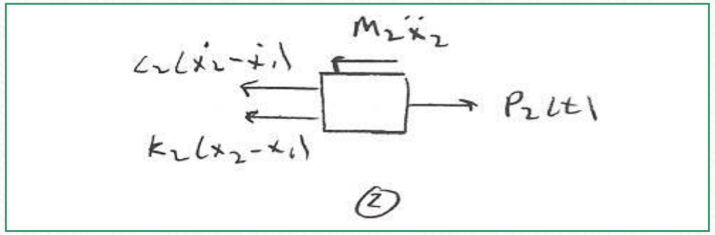


Two-DOF model problem

Matrix form of governing equation:







$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} (c_1+c_2) & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

where: [M] = mass matrix; [C] = damping matrix;

[K] = stiffness matrix; {P} = force vector

Note: Matrices have positive diagonals and are symmetric.

DISPLACEMENT TRANSMISSIBILITY

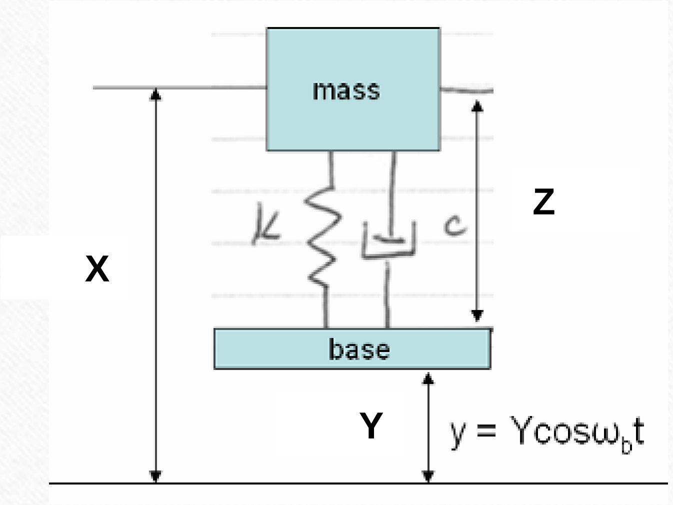
FROM BASE TO MASS IN BASE EXCITATION

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Relative / base

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

Absolute / base



Amplification of displacement amplitude (transmissibility of displacement amplitude)

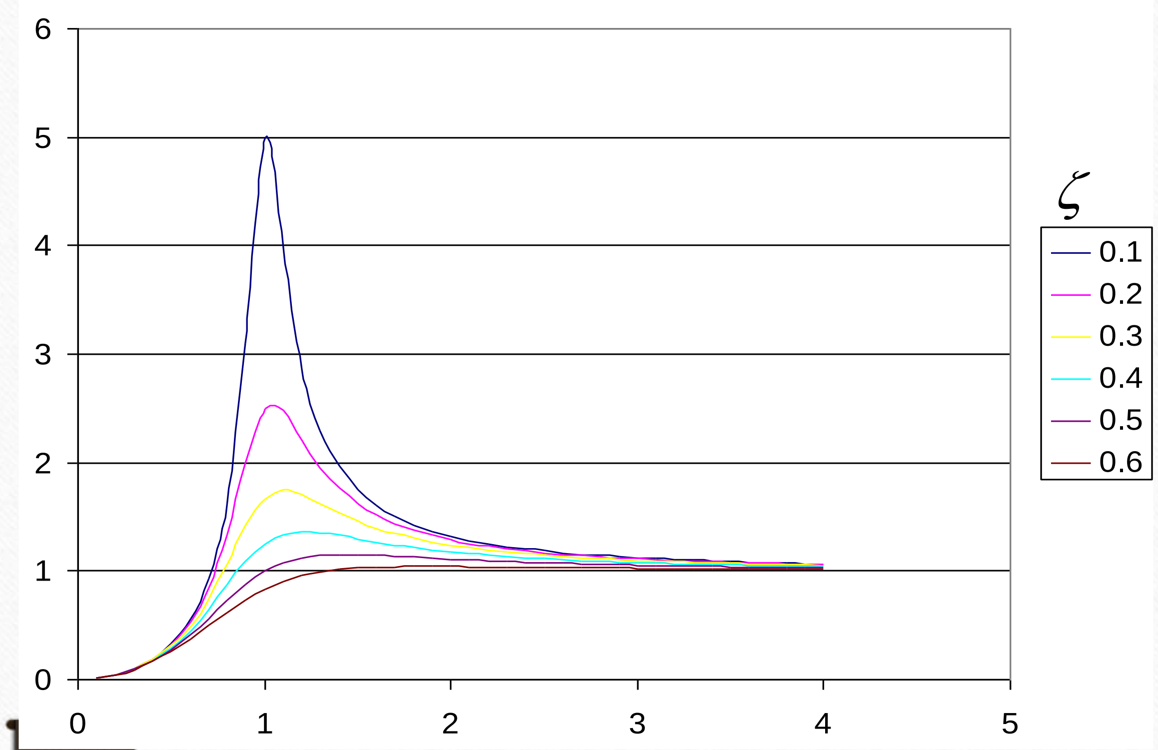
VIBRATION MEASURING DEVICES - SEISMOMETER

For larger values of r the relative displacement and the displacement of the base have the same amplitude. Hence the device can be used to measure base displacement if the frequency of base displacement is at least three time the device natural frequency

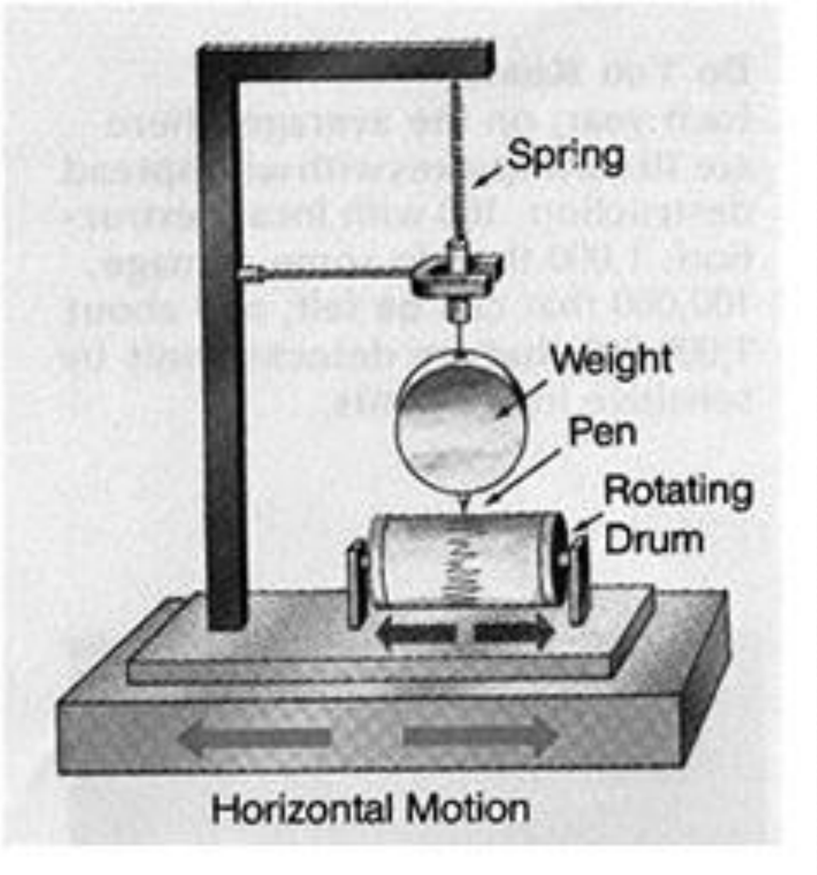
$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

When r $\rightarrow \infty$

$$\lim_{r\to\infty} \frac{\text{relative displacement Z}}{\text{base displacementY}} = 1$$



VIBRATION MEASURING DEVICES - SEISMOMETER



Seismometer Measures base displacement

VIBRATION MEASURING DEVICES - ACCELEROMETER

What happens when r → 0 **?**

Solution of the relative displacement

$$z(t) = \frac{\omega_b^2 Y}{\sqrt{(\omega_n^2 - \omega_b^2)^2 + (2\zeta\omega_n\omega_b)^2}} \cos\left[\omega_b t + \left(-\tan^{-1}\frac{2\zeta\omega_n\omega_b}{\omega_n^2 - \omega_b^2}\right)\right]$$

can be presented as

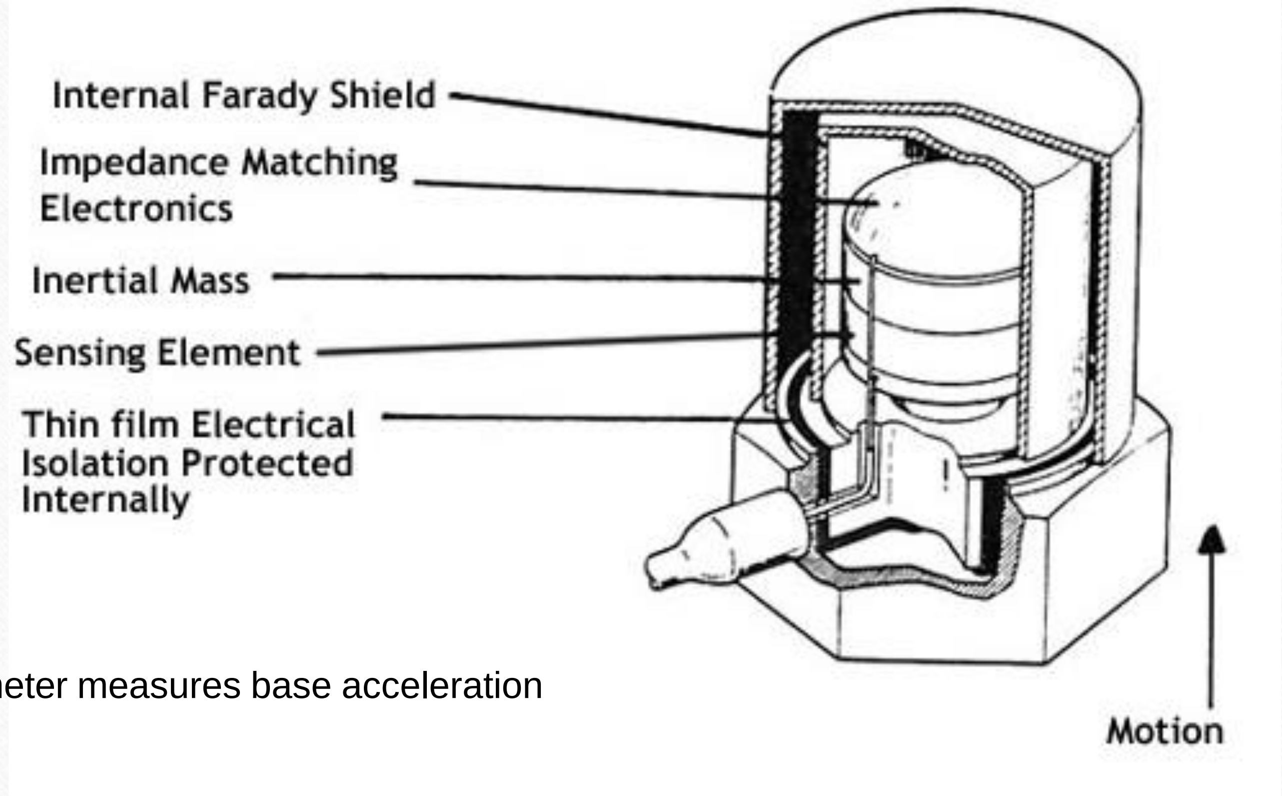
$$\omega_n^2 z(t) = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \omega_b^2 Y \cos(\omega_b t - \theta)$$

$$\omega_n^2 z(t) = \frac{-1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \ddot{y}(t)$$

$$\lim_{r\to 0} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1$$

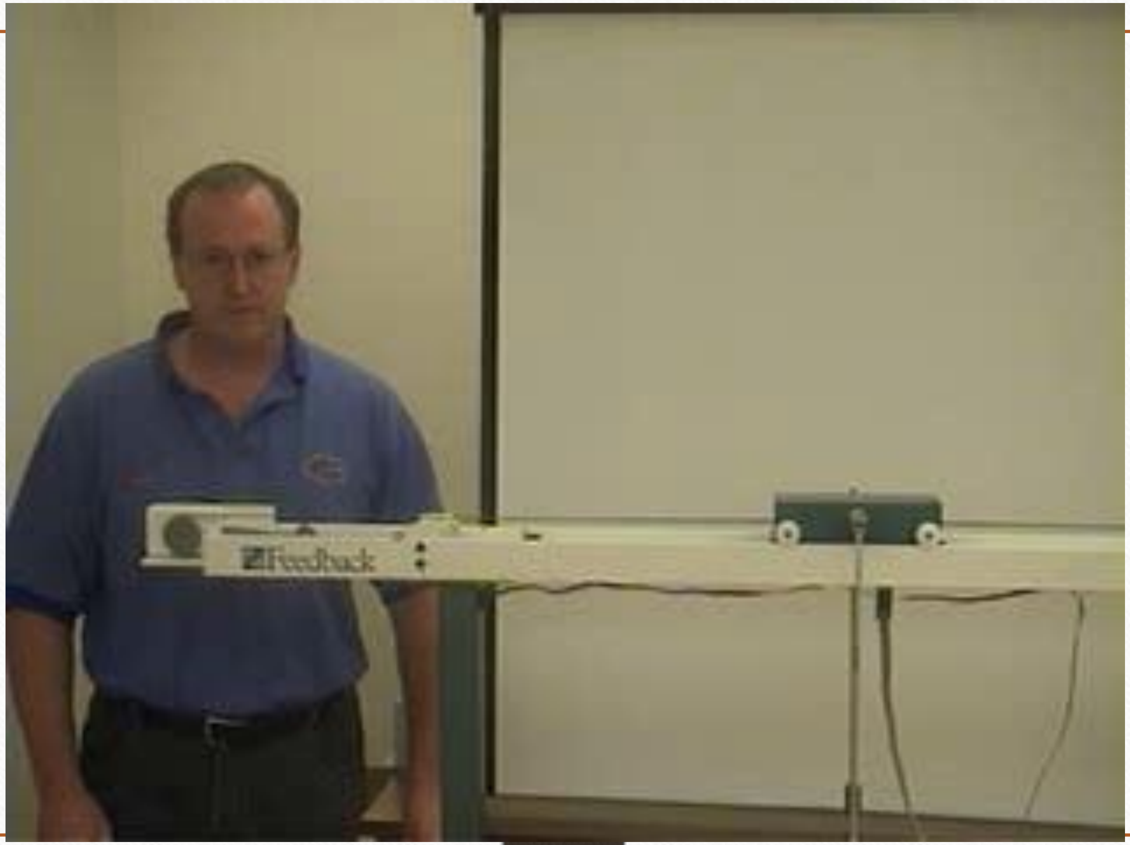
therefore $\omega_n^2 z(t)$ is proportional to base acceleration $\ddot{y}(t)$

VIBRATION MEASURING DEVICES - ACCELEROMETER

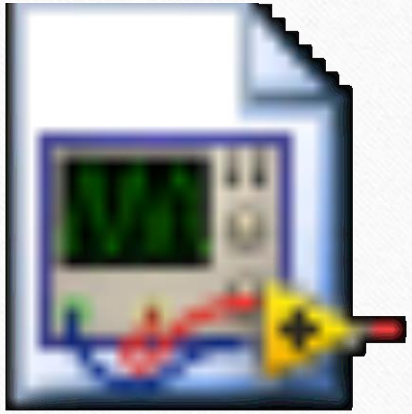


Accelerometer measures base acceleration

Inverted Pendulum



Program



Express Vibration Lab.vi